130 and the Cube Spectrum

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Abstract: The asymptotic mean degeneracy and spacing of the eigenvalue spectrum of the wave equation for a cube-shaped domain is presented. This could be achieved by realizing the following new property of the number 130. It is conjectured that there exist just 10 numbers, which can neither be represented as (i) a sum of three positive squares, nor as (ii) $4^a(8b+7)$, and which (iii) are not multiples of 4. 130 is the largest of these 10 numbers. This conjecture was checked numerically for numbers up to 40.000.

1. Motivation

Consider the inner Dirichlet problem for a cube shaped domain $G \subset R^3$. The oscillations U obey the scalar wave equation

$$\nabla^2 U + k^2 U = 0 \qquad (1)$$

and satisfy the boundary condition

$$U = 0 \qquad (2)$$

on ∂G. Let A denote the edge length of the cube. The reduced eigenvalues of the problem (1), (2),

$$E = (A k/\pi)^2 \qquad (3)$$

can be written as sums of three squares of integer numbers n_i,

$$E = n_1^2 + n_2^2 + n_3^2 \qquad (4)$$

with the additional requirement

$$n_i \neq 0 \quad \forall\ i = 1, 2, 3 . \qquad (5)$$

We observe that negative n_i do not provide further linear independent solutions of (1), (2) and may be disregarded.

For this problem the mean asymptotic spectral density is well known [1],

$$\bar{D}(E)\,dE = \left(\frac{\pi}{4}E^{1/2} - \frac{3\pi}{8} + \frac{3}{8}E^{-1/2}\right)dE, \quad (6)$$

which in terms of the wave number reads

$$\bar{D}(k)\,dk = \left(\frac{1}{2\pi^2}Vk^2 - \frac{1}{8\pi}Wk + \frac{1}{8\pi^2}L\right)dk \quad (7)$$

Here $V = A^3$, $W = 6A^2$, $L = 6\pi A$ are the volume, the surface area, and the mean total curvature, respectively. However, the mean degeneracy $\bar{G}(E)$ and spacing $\bar{\delta}(E)$ of the eigenvalues with

$$\bar{D}(E) = \bar{G}(E)/\bar{\delta}(E) \quad (8)$$

were unknown hitherto.. It is the aim of this paper to determine the asymptotic behaviour of $\bar{G}$ and $\bar{\delta}$.

2. Results

According to (4), the mean level spacing is given by

$$\bar{\delta}(E) = (1 - \bar{\rho}(E))^{-1} \quad (9)$$

where $\bar{\rho}$ is the ratio of the non-eigenvalues among the positive integers.We find

$$\bar{\rho} = \bar{\rho}' + \bar{\rho}'' = 1/6 + (5/\log 2)E^{-1} \qquad (10)$$

The term $\bar{\rho}' = 1/6$ is due to the numbers $4^a(8b + 7)$, which are not representable as a sum of three squares, and was derived by E. Landau[2] as early as 1908. Our refinement $\bar{\rho}''$ is due to those numbers h which are non-eigenvalues, because they can be represented as a sum of three squares if and only if at least one of the n_i is zero.

The structure of the set $\mathcal{H} = \{h\}$ is studied below. For $h \leqq 40\,000$ it is found by computation[3] that each h can be represented as

$$h = 4^a \cdot B_i \qquad (11)$$

where the B_i are "basic numbers". The main result is that only ten numbers $B_i > 0$ exist, the largest of them being

$$B_{10} = 130. \qquad (12)$$

From (11) the ratio $\bar{\rho}''$ is derived (see section 6 below).

Thus from (9) and (10) we obtain the asymptotic mean <u>level spacing</u>

$$\bar{\delta} = 6/5 + (36/5\log 2)E^{-1} + \mathcal{O}(E^{-2}) \qquad (13)$$

and from (6), (8), and (13) we determine the asymptotic mean degeneracy

$$\bar{G} = \frac{3\pi}{10} E^{1/2} - \frac{9\pi}{20} + \frac{9}{10}\left(\frac{1}{2} + \frac{2\pi}{\log 2}\right) E^{-\frac{1}{2}} + \mathcal{O}(E^{-1}) \tag{14}$$

for the eigenvalue spectrum of a cube.
We remind that the harmonic oscillator is the only case hitherto where the asymptotic behaviour of the spectrum is known in such detail.

3. Definitions of $\mathcal{H}$

The eigenvalues (4) are connected with the integer numbers representable by the sum of three squares. However, those numbers E have to be eliminated which can be represented as a sum of three squares if and only if at least one of the n_i's is zero. This particular set $\mathcal{H}$ of non negative integers is the subject of the following study.

Let us denote by N the set of positive integers, $N = \{1, 2, 3, \ldots\}$, and by M the set of non-negative integers, $M = N \cup \{0\}$. It is well-known that

$$E \neq 4^{a}(8b + 7) \tag{15}$$

is a necessary and sufficient condition for E to be representable by three squares n_i^2, $n_i \in M$, where a,

$b \in M$. This is the famous three-square theorem due to Legendre[4]. Thus, we obtain the

Definition 1:

$$\mathcal{H} = \left\{ h \;\middle|\; h \neq \ell^2 + m^2 + n^2;\ \ell, m, n \in \mathbf{N};\ h \neq 4^a (8b+7);\ a, b \in M \right\} \qquad (16)$$

We observe that this definition is particularly useful for the computer evaluation of $\mathcal{H}$.

An equivalent definition of $\mathcal{H}$ can easily be given in terms of diophantine equations: Any positive $h \in \mathcal{H}$ is a square or a sum of two squares such that the diophantine equations

$$h \equiv i^2 = \ell^2 + m^2 + n^2 \qquad (17)$$

and

$$h \equiv i^2 + j^2 = \ell^2 + m^2 + n^2 \qquad (18)$$

respectively, have no solution in $i, j, \ell, m, n \in \mathbf{N}$. E. g. $9 = 3^2 = 2^2 + 2^2 + 1^2$ and $17 = 4^2 + 1^2 = 3^2 + 2^2 + 2^2$ are not elements of $\mathcal{H}$. In other words, we obtain the equivalent

Definition 2:

$$\mathcal{H} = \{ h \mid h = a^2 + b^2 ; \; a, b \in M ; \\ h \neq l^2 + m^2 + n^2 ; \\ l, m, n \in N \} . \tag{19}$$

Because of the Legendre three-square-theorem, the above definitions are equivalent to the more elegant

Definition 3:

$$\mathcal{H} = \{ h \mid h = n_1^2 + n_2^2 + n_3^2 \\ \Rightarrow n_1 \cdot n_2 \cdot n_3 = 0 \} \tag{20}$$

4. Computer Results for $h \leqq 40\ 000$

We searched elements $h \in \mathcal{H}$ among the numbers $n \in N$ with $n \leqq 40.000$ by means of a computer program based on definition 1. We had to consider approximately 4.000.000 triples (n_1, n_2, n_3). We found the first 63 elements of $\mathcal{H}$, which we ordered according to

$$0 = h_1 < h_2 < h_3 < \ldots . \tag{21}$$

These h_n are listed in table 1.

5. Conjectures on the Structure of $\mathcal{H}$

The above computer results lead to several conjectures concerning the structure of $\mathcal{H}$. First of all, we notice that the number n of elements $h_n \in \mathcal{H}$ not exceeding a certain limit h, is increasing rather slowly:

Conjecture 1:

$$n(h) = \sum_{h_n \leqq h} 1 \quad \sim \quad O(\log h) \tag{22}$$

as $h \to \infty$.

This implies the following conjectures on the number of solutions of the diophantine equations (17) and (18).
Let v_n and w_n be solutions of (17) and (18), respectively.
Consider the numbers of solutions,

$$n(v) = \sum_{v_n \leqq v} 1 \quad , \quad n(w) = \sum_{w_n \leqq w} 1 \tag{23}$$

not exceeding a certain limit. Then we obtain the

Conjecture 2:

$$n(v) \sim O(v^{1/2}) \quad \text{as} \quad v \to \infty \tag{24}$$

$$n(w) \sim O(w) \quad \text{as} \quad w \to \infty \tag{25}$$

because the number of the $v = i^2$ and the $w = i^2 + j^2$ *not* solving (17) and (18) respectively is less than or equals $\mathcal{O}(\log v)$ and $\mathcal{O}(\log w)$ respectively. We thus may say that *nearly all squares and nearly all sums of two squares can be represented by a sum of three squares.*

It is easily noticed that the $h_n > 0$ appearing in table 1 can be written as multiples of the 11 basic numbers B_i in the table, printed in bold types. We thus are led to the

Conjecture 3:
If $h \in \mathcal{H}$ then

$$h = 4^a \cdot B_i, \quad i = 0,1,\ldots,10; \tag{26}$$

where $a \in M$ and

$$\{B_0, B_1, \ldots, B_{10}\} = \{0,1,2,5,10,13,25,37,58,85,130\}\ . \tag{27}$$

It is an interesting question whether these basic numbers are sufficient for the generation of *all* h by multiplication with a power of 4 or whether further $B_i > 130$ are needed. For $h < 40.000$, by our numerical computation no further B_j was found beyond 130. Thus, *130 is the largest number not representable as a sum of three positive squares, nor as $4^a\,(8\,b + 7)$, and not divisible by 4.*

For sufficiently large h, namely $h > 130$, we are able to determine h_n as a function of n by the following relations:

Conjecture 4:

$$h_{10m+a} = 4^{m-1} h_{10+a}, \quad (m \geqq 1); \tag{28}$$

$$h_{10m+b} = 4^{m-2} h_{20+b}, \quad (m \geqq 2); \tag{29}$$

with a = 3,....,9, b = 0, 1, 2.

From (28) and (29) one concludes

$$n(h) = n_0 + (10/\log 4) \cdot \log(h/h_{n_0}) \tag{30}$$

with some starting number n_0. Hence, (22) can be refined to

$$n(h) \sim (5/\log 2) \cdot \log h \,. \tag{31}$$

Of course, conjecture 4 can be derived directly from conjecture 3.

6. Conclusion

The scalar Dirichlet problem leads us to the particular set $\mathcal{H}$ of numbers representable by a sum of three squares if and only if at least one zero appears among the three squares. We computed the 63 elements $h \in \mathcal{H}$ with $h \leqq 40.000$

and found a surprisingly simple structure described by (26) - (29). Implications on the number of solutions of the diophantine equations $i^2 = \ell^2 + m^2 + n^2$ and $i^2 + j^2 = \ell^2 + m^2 + n^2$ are mentioned. The authors are well aware that one must be extremly careful with conjectures based on finite computations, and know about the $\pi(x) < li(x)$ pitfall, e. g. [5] . **Nontheless** , the structure of the $\hbar_n$ presented here seems to deserve further mathematical interest.

On the other hand, the conjectured property of 130 leads to relevant physical results, namely the mean asymptotic level spacing and degeneracy. By differentiation of (30) with respect to $\hbar$ and putting $\hbar = E$, we derive

$$\rho'' = dn(E)/dE = (10/\log 4)E^{-1} = (5/\log 2)E^{-1} \qquad (32)$$

Hence the mean level spacing is

$$\bar{\delta} = (1 - 1/6 - \rho'')^{-1} = (6/5)\left(1 - (6/\log 2)E^{-1}\right)^{-1} \qquad (33)$$

Using the asymptotic mean spectral density (6) we derive the asymptotic mean degeneracy

$$\bar{G} = \bar{D} \cdot \bar{\delta} = \frac{3}{20} \cdot \frac{2\pi E^{1/2} - 3\pi + 3E^{-1/2}}{1 - (6/\log 2)E^{-1}} \cdot \qquad (34)$$

Expanding (33) and (34) for E → ∞ we obtain the results presented in section 2.

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	3	4	5	6	7	8	9	10	11	12
-10									**0**	**1**
0	**2**	4	**5**	8	**10**	**13**	16	20	**25**	43
10	**37**	40	52	**58**	64	80	**85**	100	128	**130**
20	148	160	208	232	256	320	340	400	512	520
30	592	640	832	928	1024	1280	1360	1600	2048	2080
40	2368	2560	3328	3712	4096	5120	5440	6400	8192	8320
50	9472	10240	13312	14848	16384	20480	21760	25600	32768	33280
60	37888	...	...	...	...	....	...	...	...	...

Table 1 The elements $h \in \mathcal{H}$ with $h < 40.000$ arranged in decimal order. E.g., $h_{37} = 1024$. The basic numbers B_i are printed in bold types. Each of the above numbers can be represented in the form $h = 4^a \cdot B_i$. In particular, the (n + 1)-th row of the above array is obtained by multiplication of the n-th row by 4, if $n \geqq 3$.