Surface- and Curvature Tension of Nuclei

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Abstract:

The semi-empiric formula for nuclear masses of Weizsäcker-Bethe type should contain not only a surface-dependent but also a curvature dependent term. The coefficients can be calculated from the shell-model. In the most simple version, the Fermigas-model, the curvature term is proportional to the mean total curvature of the nucleus and to $A^{1/3}$. The theoretical estimation of the curvature tension agrees with the experimental values, determined from threshold- and scission point calculations in nuclear fission.

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I Introduction

The ground state binding energy E of the nuclei can be described approximately by a semi-empiric formula of Weizsäcker-Bethe type which consists of "liquid-drop" terms and an additional shell correction term as proposed e.g. by W.J.Swiatecki[1] or G.Süßmann[2]. If the adiabatic approximation is assumed, this formula as well can be used to calculate the binding energy of a nucleus throughout the process of nuclear fission (as well as low energy heavy nuclei fusion). But it has to be provided that the shape-dependance of the binding energy is carefully taken into account in the formula. The shape dependance can not be determined by fitting E to the about 1ooo exp. wellknown masses of stable nuclei because of their almost equal (spherical) shape. One only gets the approximate A-dependance of the liquid drop part of E which we are studying in this paper:

$$(1) \qquad E_{L.D.}(A) = a_1 \cdot A + a_2 \cdot A^{2/3} + a_3 \cdot A^{1/3} + \ldots$$

We have ommitted here the shell correction term, the pairing energy (which seems to be (to a small extent) shape-dependent) and the wellknown Coulomb term. a_1 and a_2 are of the order of 16 MeV. Usually the third term is neglected since it seems that

$$(2) \qquad a_2 \cdot A^{1/3} \gg a_3$$

for heavy nuclei. But it will prove to be important for the shape dependence study of E. Equation (1) may be regarded to be the beginning of a power series in $A^{-1/3}$. It should be noted that if one takes $a_3 \neq 0$ into account, the numerical values for the coefficients a_1, a_2, fitted with the assumption of $a_3=o$ should be renormalized. If one takes $a_1=\tilde{a}_1$, then

$$(3) \qquad \int_{A_1}^{A_2} \tilde{a}_2 \cdot A^{2/3} \cdot g(A)dA = \int_{A_1}^{A_2} dA \cdot g(A) \cdot (a_2 A^{2/3} + a_3 \cdot A^{1/3}).$$

With g(A) being the numbers of isobars, $[A_1, A_2]$ the range of the fit. Taking $A_1=o$, g(A)=const., one simply has

$$(4) \qquad a_2 = \tilde{a}_2 - \tfrac{5}{4} a_3 \cdot A_2^{-1/3} \quad .$$

The nuclear density $\varrho(\vec{x})$ has two outstanding properties: inside the heavy nuclei it is equal to the saturation value $\varrho_0 = \left(\frac{3}{4\pi}\right) \cdot r_0^{-3}$, r_0=1.123 fm., at the surface of the nucleus $\varrho(\vec{x})$ is going down to zero within a distance t=2.4 fm. t and ϱ_0 are independent of A . So in analogy to the classical liquid drop theory, one can assume that $a_1 \cdot A$ and $a_2\ A^{1/3}$ are proportional to the volume V and to the surface W respectively of the nucleus. We will show that $a_3 A^{1/3}$ is proportional to a third geometric quantity, the mean total curvature L . Now it is meaningfull, to define the surface-sheet $\mathcal{W}$ of the nucleus, using the nuclear density $\varrho(\vec{x})$. We propose the implicit definition that $\varrho(\vec{x})$ should be const. $=\varrho_w$ if $\vec{x} \in \mathcal{W}$ with the norm

$$(5) \qquad V := \frac{1}{\varrho_0} \cdot \int dV \cdot \varrho(\vec{x}) \quad = \frac{1}{\varrho_0} \cdot A \quad .$$

With $\mathcal{W}$ being defined, we can describe the shape of the nucleus in terms of the shape of $\mathcal{W}$ using the integral-definition (5) for the volume, and

$$(6) \quad V := \frac{1}{3} \int \vec{r}\, \overrightarrow{d\mathcal{W}} \quad , \quad W := \int \vec{n}\, \overrightarrow{d\mathcal{W}} \quad , \quad L := \int \left(\frac{1}{R_1} + \frac{1}{R_2}\right) \cdot \vec{n}\, \overrightarrow{d\mathcal{W}}$$

for the Surface area W and the mean total curvature L respectively, $\overrightarrow{d\mathcal{W}}$ being an oriented element of $\mathcal{W}$, $\vec{n}$ its normal vector, R_1, R_2 the main curvature radii and $\vec{x}$ the space vector of $d\mathcal{W}$. If one thinks of the surface-parallel sheets at the distance ε , defined by $\varrho(\vec{x}) = \varrho_c + \varepsilon$, these are Steiner-Sheets, for which it is wellknown that

$$(7) \qquad W = \left(\frac{\partial V}{\partial \varepsilon}\right)_{\varepsilon=0} \qquad , \quad L = \left(\frac{\partial W}{\partial \varepsilon}\right)_{\varepsilon=0} \quad .$$

In analogy one may define the surface area and the curvature of the nucleus by integrals over the volume; with (5):

$$(8) \qquad W_A := \frac{1}{\varrho_0} \cdot \int dV\, |\mathrm{grad}\, \varrho| \quad , \quad L_A := \frac{1}{\varrho_0} \cdot \int dV\, |\mathrm{div}\, \mathrm{grad}\, \varrho| \quad .$$

Finally we define the dimensionless shape-parameters:

(9) $$w := W/V^{2/3}, \quad l := L/V^{1/3}.$$

II Theory

Within the most simple quantum-mechanical model for the nucleus, the Fermigas model, we now calculate the coefficients of (1). Let us consider A free nucleons inside an external potential well $\mathcal{W}_p$, the potential being zero inside of $\mathcal{W}_p$, and infinite outside. W.J.Swiatecki[4], and P.J.Hill&J.A.Wheeler[5] calculated the binding energy of the A-nucleon system for zero temperature to be

(1o) $$E(A,\varrho_0,w_p,l_p) = \frac{\hbar^2}{2m}\cdot\left(\tfrac{3}{2}\pi^2\varrho_0\right)^{2/3}\cdot\left[\tfrac{3}{5}A + w_p\cdot A^{2/3}\cdot\left(\frac{(18\pi)^{1/3}}{80}\right) + A^{1/3}\cdot\left(\frac{4}{9\pi}\right)^{1/3}\cdot\left\{\frac{3\pi^2 w_p^2}{320} - \frac{1}{5}l_p\right\}\right] + \dots + \text{residual terms,}$$

in the special case that the potential well has the shape of a square box. The first terms can be regarded as the beginning of a power series in $\varkappa := (\tfrac{3}{2}\pi^2 A)^{-1/3} \ll 1$. For U_{235}, e.g., $\varkappa = (15.2)^{-1}$, which is sufficiently small for neglecting higher terms as $\varkappa^2$.

H.Weyl[6] proved that a_1 in (1o) is independent of the shape of the potential well. E.Hilf&G.Süßmann[7] could support the proposition that the second term too is independent of the shape, i.e. proportional to w only.

For $w_p=(36\pi)^{+1/3}$ (i.e. spherical shape) one gets for the surface term of (1o): $a_2 \cdot A^{2/3}$, a_2=9.2MeV. Using a more realistic potential (with strong velocity dependence) one gets a very good agreement[8] with the experimental value a_2=17.8MeV. For calculating the curvature tension a_3, a more detailed discussion is necessary. This is a consequence of the fact that the potential sheet only has to be regarded to be a tool in order to get the nucleon wave functions and eigenvalues which determine the binding energy and the nuclear density $\varrho(\vec{x})$, which is "felt" by scattering experiments. This was already pointed out by Swiatecki. So, in order to get the right theoretical analogon to (1) we have to transform $E(A,\varrho_0,w_p,l_p)$ into $E(A,\varrho_0,w,l)$, which means $(w_p,l_p) \rightarrow (w,l)$. If the density surface and the potential surface would have equal shape, then $w_p=w$, $l_p=l$ (the only example is the sphere). In most cases however, the density surface is more round and smooth. Since we regard the density- and potential surface to be neighboured (but not parallel), we can evaluate to the first order the potentialshape-parameters in terms of the density ones:

(11) $\quad V_\rho = V + \alpha W + \frac{1}{2}\beta^2 L \; ; \quad W_\rho = W + \gamma L \; ; \quad L_\rho = L \; .$

If density and potential surface were parallel, then $\alpha = \beta = \gamma$. From (1) we get with the definition $C := \rho_0^{1/3} \cdot A^{-1/3}$

(12) $\quad w_\rho = w - \frac{2}{3}\alpha C \cdot w^2 + \gamma C \cdot l \; ; \quad l_\rho = l \; .$

Inserting (12) into (1o) we get the final equation

(13) $\quad E(A,\rho_0,w,l) = \frac{\hbar^2}{2m}\cdot\left(\frac{3}{2}\pi^2\rho_0\right)^{2/3}\cdot\Big[\frac{3}{5}A + \frac{(18\pi)^{1/3}}{80}\cdot w\cdot A^{2/3} + (12\pi^2)^{1/3}\cdot A^{1/3}\cdot\Big\{ w^2\Big[\frac{\pi}{80} - \frac{\alpha\cdot(\frac{3}{2}\pi^2\rho_0)^{1/3}}{120\pi}\Big] + l\cdot\Big[-\frac{1}{45\pi} + \frac{\gamma\cdot(\frac{3}{2}\pi^2\rho_0)^{1/3}}{80\pi}\Big]\Big\}\Big] \; .$

The constant β does not enter (13). α and γ has to be determined by calculating V,W,and L for a given potential sheet.

Now the essential core of our argumentation is that (1o) is only valid for arbitrary shape to second order in $A^{-1/3}$. (1o) has been derived for the special case of a square box potential. So one has to calculate α and γ for the same square box potential. We make the proposition that, with $\alpha = \alpha_0$ and $\gamma = \gamma_0$ (13) is valid to third order in $A^{-1/3}$ for any shape of the nucleus. With this proposition we have estimated numerically (14 min. IBM 7o9o) curvature tension for heavy nuclei from the Fermigas model. The result is

(14) $\quad E_{kr.} = \hat{a}_3 \cdot A^{1/3} \quad , \quad 1.9 << \hat{a}_3 \approx 13.3$ MeV.

for spherical nuclear shape (γ = 2.o15 fm.). $\hat{a}_3$ is that part of (13) which is proportional to l . A more refined calculation of $\hat{a}_3$ is under investigation. But we emphazise the facts that E_{kr} is proportional to $A^{1/3}$ and to l and that E_{kr} is positive. A more refined potential will change the numerical value of a_3/l but not the dependence of $E_{kr}(l,A,\rho_0)$.

III Experiment

Experimentally the curvature energy is too small compared with the surface or volume energy to be rigorously determined by fitting (1) to the binding energies of stable nuclei. One has to make use of effects, where A remains constant, whereas w and l are changing. A good example is the nuclear fission process. Especially there are two possibilities:

1) the threshold energy of a nucleus is the difference between the bindingenergies at the groundstate and at the saddlepoint.

i.e. the minimum excitation energy needed for effective fissioning:

$$(15)\quad \Delta E_{thresh.} = \frac{\hbar^2}{2m}\left(\tfrac{3}{2}\pi^2\rho_0\right)^{4/3}\left[\frac{(18\pi)^{1/3}}{\rho_0}\cdot A^{2/3}\cdot \Delta w + (12\pi^2)^{1/3}\cdot A^{1/3}\cdot\left\{\Delta(w^2)\cdot\left[\tfrac{\pi}{30}-\frac{\alpha\cdot(\frac{3}{2}\pi^2\rho_0)^{1/3}}{40\pi}\right] + \Delta\ell\cdot\left[-\tfrac{1}{15\pi}+\frac{\delta\cdot(\frac{3}{2}\pi^2\rho_0)^{1/3}}{20\pi}\right]\right\}\right] + \Delta E_{shell}\,.$$

the volume- and the pairing term vanishes. Fitting the experimentally known about 2o thresholdenergies one gets[9)]

$$(16)\qquad \hat{a}_3 \approx 6.5 \text{ MeV},$$

which is a reasonable value. Unfortunately the fit depends on the assumtions on the shell energy term, because of $\Delta E_{shell}^{(thresh.)} = E_{shell}$(ground state), since the shell energy vanishes for the large deformation of saddle point configuration. The refined numerical calculations were made by H.v.Groote. Since l for V=const. gets a minimum for spherical shapes and is smallest for (V,w)=const., when the main curvature radii are as equal as possible at every point of the nuclear surface, the additional curvatureterm in (1) leads to saddlepointshapes which for heavy nuclei are a little bit more dumb-bell like than the sausage shaped configurations one gets without curvature energy term.

2) Scission point calculations.

If one tries to calculate the mean kinetic energies, masses and charges of the fragments of binary fission, then one usually adopts the model to calculate scissionpoint shape under the condition that the potential energy has an extremum. Then the masses and charges are determined and the intermediate Coulombenergy can be interpreted as sum of the kinetic energies of the fragments. A detailed calculation has been made by R.Schade[1o,11)], using the maximum of mean exp. kinetic energies of the fragments for adjusting the paramters. He not only gets the right mean charge distribution but also a value for the curvature energy,

$$(17)\qquad \hat{a}_3 \cong 15 \text{ MeV} \pm 8 \text{ MeV}.$$

which is in good agreement with the other approaches in view of the fact that the curvature dependence of the binding energy of nuclei is a small effct which is only important if one studies processes, where the shape of the nucleus is changing very much, such as fission or low energy heavy particle fusion. The influence of curvature effects on the full dynamical process of fission is now studied by R.W.Hasse[12)].

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